

THE COVERING DIMENSION OF A DISTINGUISHED SUBSET OF THE SPECTRUM $M(H^\infty)$ OF H^∞ AND THE ALGEBRA OF REAL-SYMMETRIC AND CONTINUOUS FUNCTIONS ON $M(H^\infty)$

RAYMOND MORTINI

ABSTRACT. We show that the covering dimension, $\dim E$, of the closure E of the interval $] - 1, 1[$ in the spectrum of H^∞ equals one. Using Suárez's result that $\dim M(H^\infty) = 2$, we then compute the Bass and topological stable ranks of the algebra $C(M(H^\infty))_{\text{sym}}$ of real-symmetric continuous functions on $M(H^\infty)$.

INTRODUCTION

In recent years the real counterparts to the classical complex function algebras $A(\mathbb{D})$, $A(K)$, $H^\infty(\mathbb{D})$ have gained a certain interest due to their appearance in control theory. These are, for example, the algebras

$$A(K)_{\text{sym}} = \{f \in C(K), f \text{ holomorphic in } K^\circ \text{ and } f(z) = \overline{f(\bar{z})} \text{ for all } z \in K\},$$

where K is a real-symmetric compact set in \mathbb{C} (that is K satisfies $z \in K \iff \bar{z} \in K$),

$$A_{\mathbb{R}}(\mathbb{D}) = \{f \in A(\mathbb{D}) : f \text{ real valued on } [-1, 1]\},$$

and

$$H_{\mathbb{R}}^\infty = H_{\mathbb{R}}^\infty(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f \text{ real valued on }] - 1, 1[\}$$

(see [16, 20, 27, 28, 33, 34]). If \mathbf{D} is the closed unit disk, then of course $A(\mathbf{D})_{\text{sym}} = A_{\mathbb{R}}(\mathbb{D})$. The main feature in the papers referenced above was to give a determination of the Bass and topological stable ranks. In addition, extension problems to invertible tuples of real-symmetric functions in several complex variables were studied in [17] for the real algebras

$$C(K)_{\text{sym}} = \{f \in C(K) : f(z_1, \dots, z_n) = \overline{f(\bar{z}_1, \dots, \bar{z}_n)}\}$$

1991 *Mathematics Subject Classification*. Primary 46J15; Secondary 46J10; 30H05; 54C40; 54F45; 55M10.

Key words and phrases. Covering dimension; spectrum for bounded analytic functions; Bass stable rank; topological stable rank; real-symmetric functions .

of complex valued continuous functions on real-symmetric compact sets K in \mathbb{C}^n .

In the present work we will determine the topological and Bass stable ranks of the algebra $C(M(H^\infty))_{\text{sym}}$ of all complex-valued continuous functions on the spectrum $M(H^\infty)$ of H^∞ that satisfy $f(z) = \overline{f(\bar{z})}$ in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Note that in view of the corona theorem, \mathbb{D} can be viewed of as a dense subset of $M(H^\infty)$. We will call $C(M(H^\infty))_{\text{sym}}$ the real-symmetric algebra associated with $C(M(H^\infty))$. Let us point out that the trace of $C(M(H^\infty))$ in \mathbb{D} is a proper subalgebra of the algebra $C_b(\mathbb{D}, \mathbb{C})$ of all bounded, continuous and complex valued functions on \mathbb{D} .

From a topological view point, the space $M(H^\infty)$ is a very bizarre space; it is a non-metrizable, connected, compact Hausdorff space of cardinal at least 2^c that is neither locally connected, nor path-connected [1]. In particular, $M(H^\infty)$ is not contractible. Its covering dimension, though, is small: it is two ([29]).

A quite difficult problem is a concrete characterization of those continuous functions on \mathbb{D} that admit a continuous extension to $M(H^\infty)$. K. Hoffman showed in his fundamental work [9] that $C(M(H^\infty))$ is the smallest uniformly closed subalgebra of $C_b(\mathbb{D}, \mathbb{C})$ that contains the (complex)-valued bounded harmonic functions. C. Bishop [2] showed that $f \in C_b(\mathbb{D}, \mathbb{C})$ has a continuous extension to $M(H^\infty)$ if and only if f is uniformly continuous with respect to the hyperbolic metric in \mathbb{D} and for every $\varepsilon > 0$ there is a Carleson contour Γ in \mathbb{D} so that f is within ε of a constant on each connected component of $\mathbb{D} \setminus \Gamma$.

Henceforth, we give a thorough discussion of the algebra $C(M(H^\infty))_{\text{sym}}$. It can be looked upon as a non trivial standard model for the classical real function algebras $C(X, \tau)$ presented for example in the monograph [12] by Kulkarni and Limaye.

1. THE ALGEBRA $C(M(H^\infty))_{\text{sym}}$

We first look at several properties of the underlying space $M(H^\infty)$ that are relevant to the study of the algebra $C(M(H^\infty))_{\text{sym}}$. Let $f \in C(M(H^\infty))$ and define f^* by $f^*(z) = \overline{f(\bar{z})}$. If $f \in H^\infty$, then $f^* \in H^\infty$ and the operation σ given by $\sigma(f) = f^*$ is an algebra involution on H^∞ . It is well known that $\{f \in H^\infty : f = f^*\}$ coincides with the algebra $H_{\mathbb{R}}^\infty$ defined above. We shall now introduce the associated involution on $C(M(H^\infty))$.

Lemma 1.1. *For $m \in M(H^\infty)$, let m^* be defined as $m^*(f) = \overline{m(f^*)}$, $f \in H^\infty$. Then $m^* \in M(H^\infty)$. Moreover, if (φ_{z_α}) is a net of point*

functionals in \mathbb{D} that converges to $m \in M(H^\infty)$, then $(\varphi_{\bar{z}_\alpha})$ is a net of point functionals in \mathbb{D} that converges to $m^* \in M(H^\infty)$.

Proof. It is obvious that m^* is additive. Moreover, m^* is homogeneous because

$$m^*(\lambda f) = \overline{m((\lambda f)^*)} = \overline{m(\bar{\lambda} f^*)} = \lambda \overline{m(f^*)} = \lambda m^*(f)$$

whenever $f \in H^\infty$. Thus $m^* \in M(H^\infty)$. Now if $\varphi_{z_\alpha} \rightarrow m$, then

$$\varphi_{\bar{z}_\alpha}(f) = f(\bar{z}_\alpha) = \overline{f^*(z_\alpha)} = \overline{\varphi_{z_\alpha}(f^*)} \rightarrow \overline{m(f^*)}.$$

□

Lemma 1.2. *Let $\tau_0 : \mathbb{D} \rightarrow \mathbb{D}$ be the involution $a \mapsto \bar{a}$. Then τ_0 admits a unique extension to a topological involution τ between $M(H^\infty)$ and itself.*

Proof. Let $\varphi_a : f \mapsto f(a)$ be the evaluation functional associated with $a \in \mathbb{D}$. For $m \in M(H^\infty)$, consider the functional m^* given above. Define τ at m by $\tau(m) = m^*$. Note that $\tau(\varphi_a) = \varphi_{\bar{a}}$. Hence $\tau : M(H^\infty) \rightarrow M(H^\infty)$ is an involution between $M(H^\infty)$ and itself. It remains to show that τ is continuous on $M(H^\infty)$. So let m_α be a net in $M(H^\infty)$ converging to m . Then for $f \in H^\infty$

$$\tau(m_\alpha)(f) = m_\alpha^*(f) = \overline{m_\alpha(f^*)} \rightarrow \overline{m(f^*)} = m^*(f) = \tau(m)(f).$$

Thus τ is a topological involution extending τ_0 . □

For a topological involution τ on a compact Hausdorff space X let

$$C(X, \tau) := \{f \in C(X, \mathbb{C}) : f(\tau(m)) = \overline{f(m)} \text{ for any } m \in X\}$$

be the classical real function algebra as given for example in [12, p. 27]. Using Lemma 1.2 above, we can now represent $C(M(H^\infty))_{\text{sym}}$ as an algebra of this type:

Corollary 1.3. *Let τ be the involution from Lemma 1.2. Then*

$$C(M(H^\infty))_{\text{sym}} = C(M(H^\infty), \tau).$$

Proof. Recall that $C(M(H^\infty))_{\text{sym}}$ was defined to be the set of all functions f in $C(M(H^\infty))$ such that $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{D}$. Let $f \in C(M(H^\infty))_{\text{sym}}$ and $m \in M(H^\infty)$. The continuity of τ implies that $g \circ \tau \in C(M(H^\infty))$ whenever $g \in C(M(H^\infty))$. If m is not point evaluation at some point in \mathbb{D} then, by the corona theorem, we choose a net z_α in \mathbb{D} such that $\varphi_{z_\alpha} \rightarrow m$. Then, by Lemma 1.2

$$f(\tau(m)) = \lim f(\tau(\varphi_{z_\alpha})) = \lim f(\bar{z}_\alpha) = \lim \overline{f(z_\alpha)} = \overline{f(m)}.$$

So $C(M(H^\infty))_{\text{sym}} \subseteq C(M(H^\infty), \tau)$. The other inclusion is trivial noticing that τ restricted to \mathbb{D} is τ_0 . □

Observation 1.4. $f \in C_b(\mathbb{D}, \mathbb{C})$ has a continuous extension F to $M(H^\infty)$ if and only if f^* has.

Proof. This follows from the representation $f^* = (\overline{F \circ \tau})|_{\mathbb{D}}$ and the fact that τ is continuous on $M(H^\infty)$ (Lemma 1.2). Another way to see this is to use Hoffman's theory that states that $C(M(H^\infty))$ is the uniformly closed subalgebra of $C_b(\mathbb{D}, \mathbb{C})$ generated by bounded holomorphic functions and their complex conjugates. \square

In conformity with our previous notation, we keep on writing f^* for the function $\overline{f \circ \tau}$, whenever $f \in C(M(H^\infty))$; that is

$$f^*(m) = \overline{f(m^*)},$$

where $m \in M(H^\infty)$.

In view of Corollary 1.3 and [12, Theorem 1.3.20] we have the following result on the structure of the maximal ideals of $C(M(H^\infty))_{\text{sym}}$ and their associated multiplicative linear functionals (see also [13] for the case of the algebra $A_{\mathbb{R}}(\mathbb{D})$).

Theorem 1.5. *Let $F_\tau = \{m \in M(H^\infty) : \tau(m) = m\}$ be the set of fixed points of τ . Then the following assertions hold:*

i) An ideal I in $C(M(H^\infty))_{\text{sym}}$ is maximal if and only if

$$I = I_m := \{f \in C(M(H^\infty))_{\text{sym}} : f(m) = 0\}$$

for some $m \in M(H^\infty)$. Moreover, $I_m = I_{m^}$ for any m .*

ii) I_m has co-dimension 1 (in the real vector space $C(M(H^\infty))_{\text{sym}}$) if and only if $m \in F_\tau$;

iii) I_m has co-dimension 2 if and only if $m \in M(H^\infty) \setminus F_\tau$.

iv) The only multiplicative \mathbb{R} -linear functionals $\phi : C(M(H^\infty))_{\text{sym}} \rightarrow \mathbb{R}$ are given by $\phi(f) = f(m)$, where $m \in F_\tau$. Their kernels are those maximal ideals I_m that have co-dimension 1 in the real vector space $C(M(H^\infty))_{\text{sym}}$.

v) The remaining multiplicative \mathbb{R} -linear functionals have target space \mathbb{C} , (regarded as an algebra over \mathbb{R}), and are given by $\phi(f) = f(m)$ or $\phi(f) = \overline{f(m)}$, where $m \in M(H^\infty) \setminus F_\tau$. Their kernels are the maximal ideals I_m that have co-dimension 2 in the real vector space $C(M(H^\infty))_{\text{sym}}$.

vi) $C(M(H^\infty))$ is the complexification of $C(M(H^\infty))_{\text{sym}}$. Each $q \in C(M(H^\infty))$ can be uniquely written as $q = f + ig$, where $f, g \in C(M(H^\infty))_{\text{sym}}$. Here $f = (q + q^)/2$ and $g = (q - q^*)/(2i)$.*

vii) $\sigma(f) = f^$ is a topological involution on $C(M(H^\infty))$.*

Proof. For the proof, we just note that if $\tau(m) = m$, then the evaluation functional ϕ_m on $C(M(H^\infty))_{\text{sym}} = C(M(H^\infty), \tau)$ satisfies

$$\phi_m(f) = f(m) = f(\tau(m)) = \overline{f(m)}.$$

Hence ϕ_m is real valued and so the kernel has co-dimension 1.

On the other hand, if $\tau(m) \neq m$, then there exists (by [12, Lemma 1.3.7]) a function $f \in C(M(H^\infty), \tau)$ with $f(m) = i$ and $f(\tau(m)) = -i$. Thus the evaluation functional ϕ_m is a surjection onto the real algebra \mathbb{C} ; hence its kernel has codimension 2.

The results now follow from [12, Theorem 1.3.20]). \square

That the maximal ideal spaces of $C(M(H^\infty))_{\text{sym}}$ and $C(M(H^\infty))$ can be identified also follows from the fact that if $(f_1, \dots, f_N) \in C(M(H^\infty))_{\text{sym}}^N$, then a solution to the Bézout equation $\sum_{j=1}^N q_j f_j = 1$ in $C(M(H^\infty))$ yields the solution $\sum_{j=1}^N \frac{q_j + q_j^*}{2} f_j = 1$ of the associated Bézout equation in $C(M(H^\infty))_{\text{sym}}$.

In the same way, we may identify the maximal space of the real subalgebra $H_{\mathbb{R}}^\infty$ of $C(M(H^\infty))_{\text{sym}}$ with $M(H^\infty)$. Note, however, that if m is a character of $H_{\mathbb{R}}^\infty$, then the maximal ideals $\text{Ker } m$ and $\text{Ker } m^*$ coincide even in the case where $m \neq m^*$.

In the next section we will determine the set F_τ of fixed points of τ .

2. THE CLOSURE OF THE OPEN UNIT INTERVAL IN $M(H^\infty)$

Let E be the closure of $] -1, 1[$ in $M(H^\infty)$, M^+ the closure of $\mathbb{D}^+ := \{z \in \mathbb{D} : \text{Im } z > 0\}$ in $M(H^\infty)$ and M^- the closure of $\mathbb{D}^- := \{z \in \mathbb{D} : \text{Im } z < 0\}$ in $M(H^\infty)$. Finally, $\mathbb{T}^+ = \{e^{i\theta} : 0 < \theta < \pi\}$ and $\mathbb{T}^- = \{e^{i\theta} : -\pi < \theta < 0\}$.

The goal in this section is to prove that $M^+ \cap M^- = E$ and to show that $E = F_\tau$. To this end, we need a couple of lemmas.

For $f \in C(M(H^\infty))$, we denote by $Z(f) = \{m \in M(H^\infty) : f(m) = 0\}$ the zero set of f . The set of points in $M(H^\infty)$ with non-trivial Gleason parts will be denoted as usual, by G (see [8, 9]). The pseudo-hyperbolic distance on \mathbb{D} is given by $\rho(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|$. Its extension to $M(H^\infty)$ is defined as

$$\rho(x, m) = \sup\{|f(x)| : f \in H^\infty, \|f\|_\infty \leq 1, f(m) = 0\},$$

$x, m \in M(H^\infty)$.

Observation 2.1. *Let $f \in C(M(H^\infty))$. Then $f^*(x) = \overline{f(x)}$ whenever $x \in E$.*

Proof. Let (r_α) be a net in $] - 1, 1[$ that converges to x . Then

$$f^*(x) = \lim f^*(r_\alpha) = \lim \overline{f(\bar{r}_\alpha)} = \lim \overline{f(r_\alpha)} = \overline{f(x)}.$$

□

Our subsequent results will be based on the following assertion. Recall that for a real or complex function algebra A with character space $M(A)$ a compact set $C \subseteq M(A)$ is said to be A -convex if C coincides with its A -convex hull

$$\check{C} = \{m \in M(A) : |\hat{f}(m)| \leq \max_C |\hat{f}|, \forall f \in A\},$$

where \hat{f} denotes the Gelfand transform of f .

Theorem 2.2. *Let S be a closed subset of the closure E of $] - 1, 1[$ in $M(H^\infty)$. Then S is H^∞ -convex as well as $H_{\mathbb{R}}^\infty$ -convex.*

Proof. Let $x \notin S$. Since H^∞ is separating (see [29, p. 242]), there exists $f \in H^\infty$ such that $f(x) = 0$ and $f \neq 0$ on S . We may assume that $\|f\|_\infty \leq 1$. Let $g = ff^*$. Then $g \in H_{\mathbb{R}}^\infty$ and so g is real valued on E . By definition of g , we actually have that $1 \geq g \geq 0$ on E . Since $f \neq 0$ on S , we obtain from $f^* = \bar{f}$ on E , that $\sigma := \min_S g > 0$. Now let $h = 1 - g$. Then $h \in H_{\mathbb{R}}^\infty \subseteq H^\infty$, $h(x) = 1$ and so

$$\max_S |h| = \max_S h \leq 1 - \sigma < |h(x)|.$$

Thus x does not belong to the $H_{\mathbb{R}}^\infty$ -convex closure of S . Therefore S is $H_{\mathbb{R}}^\infty$ -convex as well as H^∞ -convex. □

Lemma 2.3. *Let b be an interpolating Blaschke product all of whose zeros z_n in \mathbb{D} satisfy $\text{Im } z_n < 0$. Suppose that $Z(b) \cap E = \emptyset$. Then $Z(b) \cap M^+ = \emptyset$ and $Z(b^*) \cap M^- = \emptyset$.*

Proof. Let $x \in M(H^\infty)$ satisfy $b(x) = 0$. We may assume that $x \notin \mathbb{D}$. Then $x \in G$ and x belongs to the closure of the $\{z_n : n \in \mathbb{N}\}$ (see [8, p. 379]). Assuming that $x \in M^+ = \text{cl}(\mathbb{D}^+)$, we get from Hoffman's result [9, p. 103] that $\rho(Z(b) \cap \mathbb{D}, \mathbb{D}^+) = 0$. For $j \in \mathbb{N}$, let $u_j \in \mathbb{D}^+$ and $n(j)$ be chosen so that $\rho(z_{n(j)}, u_j) \leq 1/j$. Then every cluster point m of $\{z_{n(j)} : j \in \mathbb{N}\}$ belongs to $Z(b)$.

Note that $\text{Im } z_{n(j)} < 0$ and $\text{Im } u_j > 0$. By passing to subnets, we may assume that $z_{n(j(\alpha))} \rightarrow m$. Now, if $\text{Im } a < 0$ and $\text{Im } \xi \geq 0$, then

$$\rho(a, \text{Re } a) \leq \rho(a, \bar{a}) \leq \rho(a, \xi) + \rho(\xi, \bar{a}) \leq 2\rho(a, \xi).$$

Now letting $a = z_{n(j(\alpha))}$ and $\xi = u_{j(\alpha)}$, we obtain that

$$\rho(z_{n(j(\alpha))}, \text{Re } z_{n(j(\alpha))}) \rightarrow 0.$$

By taking a further subnet, if necessary, $\operatorname{Re} z_{n(j(\beta))}$ then converges to some m' . Note that this implies that $m' \in E$. Since ρ is semi-continuous [9, p. 103], $\rho(m, m') = 0$ and so $m = m'$. Thus $m \in E \cap Z(b)$. Hence $Z(b) \cap E \neq \emptyset$; a contradiction to our hypothesis. Therefore $x \notin M^+$. Since x was an arbitrary zero of b , we conclude that $Z(b) \cap M^+ = \emptyset$. Due to symmetry, we obviously have that $Z(b^*) \cap M^- = \emptyset$, too. \square

Let us note that the previous result also holds for arbitrary Blaschke products (see Proposition 2.9 at the end of this section). For the sake of completeness we present that result, too, although we will not use this fact in the present paper. The proof itself is based on Theorem 2.6 and on a factorization theorem given by K. Izuchi.

Versions of the following function theoretic lemma are well known. What we need here, are uniform estimates outside some cones. For the reader's convenience we present its proof.

Lemma 2.4. *Let u be the harmonic function with boundary values 1 on \mathbb{T}^+ and 0 on \mathbb{T}^- and let C_κ be the cone*

$$C_\kappa = \{z = x + iy \in \mathbb{D} : |y| \leq \kappa(1 - x)\}.$$

Then there exists $\sigma > 0$ such that $1 > u(z) \geq 3/4$ on

$$h^+(C_\sigma) := \{z = x + iy \in \mathbb{D}, 0 \leq \sigma(1 - x) \leq y\}$$

and $0 < u(z) \leq 1/4$ on

$$h^-(C_\sigma) := \{z = x + iy \in \mathbb{D}, y < 0, 0 \leq \sigma(1 - x) \leq |y|\}.$$

Moreover, $u(r) = 1/2$ for every $r \in]-1, 1[$, $1/2 \leq u \leq 1$ on \mathbb{D}^+ and $0 \leq u \leq 1/2$ on \mathbb{D}^- .

Proof. Note that u has the form

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^\pi \frac{1 - r^2}{1 + r^2 - 2r \cos(t - \theta)} dt.$$

Now we use the following inequality:

$$1 + r^2 - 2r \cos s = (1 - r)^2 + 4r \sin^2(s/2) \leq (1 - r)^2 + s^2.$$

Let $\theta \in [0, \pi/4]$. Then

$$\begin{aligned} u(re^{i\theta}) &\geq \int_0^\theta + \int_\theta^{\pi/2} \geq \\ &\frac{1+r}{1-r} \frac{1}{2\pi} \int_0^\theta \frac{1}{1 + \left(\frac{\theta-t}{1-r}\right)^2} dt + \frac{1+r}{1-r} \frac{1}{2\pi} \int_\theta^{\pi/2} \frac{1}{1 + \left(\frac{t-\theta}{1-r}\right)^2} dt = \end{aligned}$$

$$-\frac{1+r}{2\pi} \arctan\left(\frac{\theta-t}{1-r}\right) \Big|_0^\theta + \frac{1+r}{2\pi} \arctan\left(\frac{t-\theta}{1-r}\right) \Big|_\theta^{\pi/2} =$$

$$\frac{1+r}{2\pi} \left[\arctan\left(\frac{\theta}{1-r}\right) + \arctan\left(\frac{\frac{\pi}{2}-\theta}{1-r}\right) \right].$$

Now if $re^{i\theta}$ stays outside the cone

$$\mathcal{C} := \{z = \tilde{r}e^{i\tilde{\theta}} \in \mathbb{D} : |\tilde{\theta}| < C(1-\tilde{r})\},$$

then $C(1-r) \leq \theta \leq \pi/4$ and hence

$$u(re^{i\theta}) \geq \frac{1+r}{2\pi} \left[\arctan C + \arctan\left(\frac{\pi/4}{1-r}\right) \right] \xrightarrow{r \rightarrow 1} \frac{\arctan C}{\pi} + \frac{1}{2}.$$

Now $\mathcal{C} \subseteq C_\kappa$ with $\kappa = C = \tan \psi$, where ψ is the angle between the horizontal axis and the line $y = \kappa(1-x)$ respectively the curve $r(\theta) = 1 - \frac{\theta}{C}$ at the point 1.

Thus $u(re^{i\theta}) \geq 3/4$ whenever C is large, $r = r(C)$ close to 1, and $re^{i\theta} \in h^+(C_\sigma)$ for some $\sigma = \sigma(C)$.

A change of variable $-t \rightarrow s$ shows that $u(z) + u(\bar{z})$ is the integral over the Poisson kernel on the whole interval $[0, 2\pi]$. Hence $u(z) + u(\bar{z}) = 1$. Now if $z \in h^+(C_\sigma)$, then $\bar{z} \in h^-(C_\sigma)$ and so

$$u(\bar{z}) = 1 - u(z) \leq 1 - 3/4 = 1/4.$$

Finally, if $z = x$ is real, then $1 = u(z) + u(\bar{z}) = 2u(x)$; and so $u(x) = 1/2$.

Next let $z \in \mathbb{D}^-$; that is $z = re^{i\theta}$ with $-\pi < \theta < 0$. Then $t - \theta \geq t$ and so

$$u(re^{i\theta}) \leq \frac{1}{2\pi} \int_0^\pi \frac{1-r^2}{1+r^2-2r\cos(t)} dt = u(r) = 1/2.$$

If $z \in \mathbb{D}^+$, then $u(z) = 1 - u(\bar{z}) \geq 1/2$. □

Corollary 2.5. *Let u be the harmonic function above. Then*

- (1) $1/2 \leq u \leq 1$ on M^+ ;
- (2) $0 \leq u \leq 1/2$ on M^-
- (3) $u = 1/2$ on E .

Question: do we have that $u = 1/2$ exactly on E ?

Recall that for $\lambda \in \mathbb{T}$, the fiber M_λ is given by

$$M_\lambda = \{m \in M(H^\infty) : m(z) = \lambda\},$$

where z denotes the identity function here.

Theorem 2.6. $M^+ \cap M^- = E$.

Proof. First we note that $E \subseteq M^+ \cap M^-$ by definition. Now let $y \in M^+ \cap M^-$. Of course, we may assume that $y \notin \mathbb{D}$, since the fact that

$$M^+ \cap M^- \cap \mathbb{D} =]-1, 1[$$

is obvious. We claim that y belongs to one of the two fibers M_1 or M_{-1} . In fact, suppose that $y \in M_\lambda$, where $\lambda \notin \{-1, 1\}$. We may assume that $\text{Im } \lambda > 0$. Let $p_\lambda(z) = (1 + \bar{\lambda}z)/2$ be a peak function in $A(\mathbb{D})$ associated with λ . Then $p_\lambda \equiv 1$ on $M_\lambda \subseteq M^+$, but $|p_\lambda| \leq 1 - \eta < 1$ outside small neighborhoods of λ within \mathbf{D} . In particular $|p_\lambda| \leq 1 - \eta$ on M^- . Hence $y \notin M^+ \cap M^-$; a contradiction.

Let $x \in M(H^\infty) \setminus E$. Since $M^+ \cup M^- = M(H^\infty)$, we may assume that $x \in M^+$. Also, by the paragraph above, we may assume that $x \in M_1$. We claim that x does not belong to M^- .

Case 1 $x \in G$.

Choose a closed neighborhood U of x in $M(H^\infty)$ so that $U \cap E = \emptyset$. There exists an interpolating Blaschke product b with $b(x) = 0$ such that $Z(b) \cap \mathbb{D} \subseteq U^\circ \cap \mathbb{D}$. Thus, by [8, p.379], $Z(b) \subseteq U$. Hence $Z(b) \cap E = \emptyset$. Decompose b in a product $b = b_1 b_2$ of two interpolating Blaschke products, where the zeros of b_1 are those with imaginary part strictly positive and where the zeros of b_2 are those with imaginary part strictly negative. Note that b has no real zeros.

Noticing that $Z(b_2) \cap E = \emptyset$, we obtain from Lemma 2.3 that $Z(b_2) \cap M^+ = \emptyset$. Thus $b_1(x) = 0$. Again, since $Z(b_1) \cap E = \emptyset$ and the zeros of b_1 are contained in M^+ , we obtain from Lemma 2.3, that $Z(b_1) \cap M^- = \emptyset$. Thus $x \notin M^-$. Hence, $M^+ \cap M^- \cap G \subseteq E$.

Case 2 x is a trivial point with $x \in M^+ \cap M_1$.

Since the $M(H^\infty)$ -closure S of every cone

$$C_\kappa = \{z = \xi + i\eta \in \mathbb{D} : \xi \geq 1/2, |\eta| \leq \kappa(1 - \xi)\}$$

is contained in G (see [9, p. 108]), x is not in S . By Lemma 2.4, the harmonic function u given there is bigger than $3/4$ on $h^+(C_\kappa)$ and smaller than $1/4$ on $h^-(C_\kappa)$ if C_κ has a sufficiently large opening. Now $\overline{h^+(C_\kappa)} \subseteq M^+$, $\overline{h^-(C_\kappa)} \subseteq M^-$, $\overline{h^+(C_\kappa)} \cap \overline{h^-(C_\kappa)} = \emptyset$ and

$$M_1 = (\overline{h^+(C_\kappa)} \cap M_1) \cup (S \cap M_1) \cup (\overline{h^-(C_\kappa)} \cap M_1).$$

Since $x \in M^+$, $u \geq 1/2$ on M^+ and $u \leq 1/4$ on $\overline{h^-(C_\kappa)}$, we deduce that $x \in h^+(C_\kappa)$ and so $u(x) \geq 3/4$. Also, since $u \leq 1/2$ on M^- , we conclude that $x \notin M^-$.

To sum up, we have shown that $(M^+ \setminus E) \cap M^- = \emptyset$. Therefore $M^+ \cap M^- = E$. \square

Corollary 2.7. *Let $m \in M(H^\infty)$. The following assertions hold:*

- (1) $m \in M^+$ if and only if $m^* \in M^-$.
- (2) The set E coincides with the set of fixed points F_τ of τ ; that is $m = m^*$ if and only if $m \in E$.
- (3) $m \in E$ if and only if $\overline{m(f^*)} = m(f)$ for any $f \in H^\infty$.

Proof. (1) Let $m \in M^+$. By Lemma 1.1, if $z_\alpha \rightarrow m$, $\text{Im } z_\alpha > 0$, then $\bar{z}_\alpha \rightarrow m^*$. Thus $m^* \in M^-$.

(2) Let $m = m^*$. By (1), $m \in M^+ \cap M^-$. Using Theorem 2.6, we conclude that $m \in E$. To prove the converse, let $m \in E$. Choose a net $r_\alpha \in]-1, 1[$ converging to m . Then, by Lemma 1.1, $r_\alpha = \bar{r}_\alpha$ converges to m^* . Thus $m = m^*$. Since $m^* = \tau(m)$, it follows that $F_\tau = E$.

(3) This is merely a reformulation of the assertion that $m = m^*$. \square

We add the following additional information on u .

Proposition 2.8. *The following assertions hold:*

- (1) $u \equiv 1$ on the set of trivial points in M^+ ;
- (2) $u \equiv 0$ on the set of trivial points in M^- .

Proof. If $x \in M^+ \cap M_\lambda$, $\lambda \in \mathbb{T} \setminus \{-1, 1\}$, then $u(x) = 1$ because u is constant 1 on M_λ . If x is a trivial point in $M_1 \cap M^+$, then x lies outside the closure of any cone. Hence we can replace the number $3/4$ in Lemma 2.4 by any number $\sigma < 1$ close to one. Thus $u(x) \geq \sigma$ and so, $u(x) = 1$. A similar reasoning holds for $x \in M^-$. \square

Using Theorem 2.6. we may generalize Lemma 2.3 in the following way.

Proposition 2.9. *Let B be a Blaschke product all of whose zeros z_n in \mathbb{D} satisfy $\text{Im } z_n < 0$. Suppose that $Z(B) \cap E = \emptyset$. Then $Z(B) \cap M^+ = \emptyset$ and $Z(B^*) \cap M^- = \emptyset$.*

Proof. The hypothesis $Z(B) \cap E = \emptyset$ and the fact that $M^+ \cap M^- = E$ (Theorem 2.6) imply that $M^+ \cap Z(B)$ and $M^- \cap Z(B)$ are disjoint, open-closed sets in $Z(B)$. Suppose that both sets are nonempty. Then, by [10, Theorem 2.1], $B = B^+ B^-$, where $Z(B^+) = M^+ \cap Z(B)$, $Z(B^-) = M^- \cap Z(B)$. But B , and hence B^+ , has no zeros in \mathbb{D}^+ . Thus the factor B^+ does not exist. This contradiction shows that $Z(B) \cap M^+ = \emptyset$. \square

Finally we remark, that if B is a Blaschke product all of whose zeros z_n in \mathbb{D} satisfy $\text{Im } z_n < 0$, then $Z(B) \cap M^+$ can be big, though. Just take the zeros z_n in \mathbb{D}^- with $\rho(z_n, 1 - \frac{1}{n^2}) \leq 1/n$ and let B be the associated Blaschke product. Then $B(r) \rightarrow 0$ as $r \rightarrow 1$ and so B vanishes identically on every Gleason part $P(x)$ associated with a point $x \in E \setminus \mathbb{D}$. We claim that, $(\overline{P(x)} \cap M^+) \setminus E \neq \emptyset$ and $(\overline{P(x)} \cap M^-) \setminus E \neq \emptyset$. In fact, suppose that $r_\alpha \rightarrow x$, $r_\alpha \in]0, 1[$. Then $w_\alpha := \frac{r_\alpha + z}{1 + r_\alpha z} \rightarrow L_x(z)$,

$w_\alpha \in \mathbb{D}^+$ for $z \in \mathbb{D}^+$, $\overline{w_\alpha} \rightarrow (L_x(z))^*$ and $\overline{w_\alpha} \rightarrow L_x(\bar{z})$. Since the Hoffman map L_x is a bijection, $L_x(\bar{z}) \neq L_x(z)$, and so $L_x(z) \neq (L_x(z))^*$. Thus, by Corollary 2.7, $L_x(z) \in (P(x) \cap M^+) \setminus E$ and $L_x(\bar{z}) \in (P(x) \cap M^-) \setminus E$. In particular, $\emptyset \neq P(x) \cap M^+ \subseteq Z(B)$.

Such a phenomenon does not occur when b is an interpolating Blaschke product, since $Z(b) \subseteq M^-$ whenever the zeros in \mathbb{D} are in the lower half-disk. Thus the fact that $M^+ \cap M^- = E$ implies that no point in $M^+ \setminus E$ can be a zero of b .

3. THE COVERING DIMENSIONS OF E AND M^+

First let us recall the definition of the notion of covering dimension (or Čech-Lebesgue dimension) as given in [6, p. 54] or [22, p. 111]. Let X be a normal topological space. Then X is said to have dimension n , denoted by $\dim X = n$, if n is the smallest integer such that every finite open covering of X has a finite open refinement of order n . Here, as usual, the order of a family \mathcal{A} of subsets of X is the largest integer n such that \mathcal{A} contains $n + 1$ sets with a non-empty intersection.

In order to determine the covering dimension of E , we need the following result from [22, p. 119]. Recall that a closed set C separates two disjoint closed sets E and F in a normal space X if $X \setminus C = G \cup H$, where G and H are two disjoint open sets with $E \subseteq G$ and $F \subseteq H$.

Proposition 3.1. *If X is a normal space, the following assertions are equivalent:*

- (1) $\dim X \leq n$;
- (2) *For each family of $n + 1$ pairs of closed sets*

$$\{(E_1, F_1), \dots, (E_{n+1}, F_{n+1})\}$$

where $E_i \cap F_i = \emptyset$, there exists a family $\{C_1, \dots, C_{n+1}\}$ of closed sets such that C_i separates E_i and F_i and $\bigcap_{i=1}^{n+1} C_i = \emptyset$.

Theorem 3.2. *a) Let E be the closure of $] - 1, 1[$ in $M(H^\infty)$. Then the covering dimension of E is one.*

b) The covering dimension of the closure, M^+ , of $\{z \in \mathbb{D} : \text{Im } z > 0\}$ in $M(H^\infty)$ is two.

Proof. a) For $j = 1, 2$, let (E_j, F_j) be two pairs of disjoint closed sets in E . By Theorem 2.2, the sets $E_j \cup F_j$ are H^∞ -convex. So the maximal ideal space of the algebras $A_j = \overline{H^\infty|_{E_j \cup F_j}}$ equals $X := E_j \cup F_j$. Since E_j and F_j are open-closed in X , Shilov's idempotent theorem (see for example [7, p. 88]), yields a function $q_j \in A_j$ such that $q_j \equiv 1$ on F_j and $q_j \equiv 0$ on E_j . Thus there exists $f_j \in H^\infty$ such that $f_j \sim 1$ on F_j and $f_j \sim 0$ on E_j . Let $h_j = f_j f_j^*$. Then $h_j \in H^\infty_{\mathbb{R}}$ and h_j is real

valued on $] - 1, 1[$, hence on E . Moreover, since for $x \in E$ one has $f^*(x) = \overline{f(x)}$ (2.1), h_j is close to 1 on F_j and close to 0 on E_j . Let $k_j = 2h_j - 1$. Then $k_j \in H_{\mathbb{R}}^{\infty}$ is real valued on E , too, and k_j is close to 1 on F_j and close to -1 on E_j . Consider the pair (k_1, k_2) . Since $H_{\mathbb{R}}^{\infty}$ has the topological stable rank 2 ([20]),¹ there is an invertible pair (g_1, g_2) of functions in $H_{\mathbb{R}}^{\infty}$ so that g_j and k_j stay very close to each other. In particular, the g_j are real valued on E and g_j remains close to -1 on E_j and close to 1 on F_j . But $Z(g_1) \cap Z(g_2) = \emptyset$. Thus we may choose $C_j = Z(g_j) \cap E$ to conclude that C_j separates E_j and F_j , (just take $G_j = \{x \in E : g_j < 0\}$ and $H_j = \{x \in E : g_j > 0\}$.) Hence, by Proposition 3.1, the covering dimension of E is less than or equal to one. The dimension cannot be zero, though, since E is a continuum. Thus $\dim E = 1$.

b) The fact that the covering dimension of the closure, M^+ , of $\{z \in \mathbb{D} : \operatorname{Im} z > 0\}$ in $M(H^{\infty})$ is two follows from Suárez's result [29] that $\dim M(H^{\infty}) = 2$ and the sum-property for the dimension [6, p.42, Theorem 1.5.3] that tells us that if X is the union of a finite (or countably infinite) number of closed sets X_j with $\dim X_j \leq d$, then $\dim X \leq d$. Here we have $X = M^+ \cup M^-$ and, due to symmetry, $\dim M^+ = \dim M^-$. \square

Instead of using in the above proof the full power of the fact that $\operatorname{tsr} H_{\mathbb{R}}^{\infty} = 2$, we can also prove part a) of Theorem 3.2 by applying the following Lemma.

Lemma 3.3. *Let (k_1, k_2) be a pair of functions in $H_{\mathbb{R}}^{\infty}$. Then, for every $\varepsilon > 0$, there exists a pair $(b_1 K_1, b_2 K_2)$ of functions in $H_{\mathbb{R}}^{\infty}$ such that*

- (1) *the b_j are interpolating Blaschke products having only real zeros;*
- (2) *b_1 and b_2 have no common zeros on $M(H^{\infty})$;*
- (3) *K_1 and K_2 are zero free on E ;*
- (4) *$\|b_j K_j - k_j\|_E < \varepsilon$.*

Proof. Let $k_j = B_j F_j$ be the Riesz factorization of k_j . Here B_j is a Blaschke product and F_j is zero free on \mathbb{D} . Since $k_j \in H_{\mathbb{R}}^{\infty}$, the zeros of B_j are symmetric to the real axis and so B_j , as well as F_j , belong to $H_{\mathbb{R}}^{\infty}$. We may assume that $F_j \geq 0$ on $] - 1, 1[$. Then for $\varepsilon > 0$, the functions $F_j + \varepsilon$ have no zeros on E . Let $B_j = v_j u_j$, where v_j is the Blaschke product formed with the real zeros of B_j . By [15], the Frostman shifts $w_j := \frac{v_j - \varepsilon}{1 - \varepsilon v_j}$ are Carleson-Newman Blaschke products. Write w_j as $w_j = d_j e_j$, where d_j is the factor of w_j formed

¹for a definition see the next section

with the real zeros. Note that $w_j, d_j, e_j \in H_{\mathbb{R}}^\infty$. Since d_j is a Carleson-Newman Blaschke product with real zeros only, it can be uniformly approximated by interpolating Blaschke products with real zeros. Let $W_j = e_j u_j$. Due to the symmetry of the zeros, $W_j(a) = 0$ if and only if $W_j(\bar{a}) = 0$. Therefore, for $r \in]-1, 1[$,

$$W_j(r) = \prod_{a: \operatorname{Im} a > 0} \frac{\bar{a}}{|a|} \frac{a-r}{1-\bar{a}r} \cdot \frac{a}{|\bar{a}|} \frac{\bar{a}-r}{1-ar} = \prod_{a: \operatorname{Im} a > 0} \frac{|a-r|^2}{|1-ar|^2}.$$

Thus $W_j \geq 0$ on $] -1, 1[$. Hence $W_j + \varepsilon$ is zero free on E . Thus we are able to approximate each k_j by functions of the form $\tilde{b}_j K_j$, where \tilde{b}_j is an interpolating Blaschke product with real zeros only and where

$$K_j = (F_j + \varepsilon)(W_j + \varepsilon).$$

Let $b_1 = \tilde{b}_1$. By moving those zeros of \tilde{b}_2 that are hyperbolically close to those of \tilde{b}_1 , we may approximate \tilde{b}_2 by an interpolating Blaschke product b_2 so that $\inf_{\mathbb{D}}(|b_1| + |b_2|) \geq \delta > 0$; for example by replacing $\tilde{b}_2 = b_2^{(1)} b_2^{(2)}$ by the interpolating Blaschke product $b_2^{(1)} \frac{b_2^{(2)} - \varepsilon}{1 - \varepsilon b_2^{(2)}}$. The tuple $(b_1 K_1, b_2 K_2)$ is now the desired item. \square

4. THE BASS AND TOPOLOGICAL STABLE RANKS FOR $C(M(H^\infty))_{\text{sym}}$

In this section we determine some K -theoretic data for the algebra $C(M(H^\infty))_{\text{sym}}$. Our construction will use the following lemma.

Lemma 4.1. *Let $q \in C(M^+, \mathbb{C})$. Suppose that q is real-valued on $] -1, 1[$. Then q admits a unique extension to $C(M(H^\infty))_{\text{sym}}$.*

Proof. Let f be defined as

$$f(m) = \begin{cases} q(m) & \text{if } m \in M^+, \\ \overline{q(m^*)} & \text{if } m \in M^-. \end{cases}$$

Since $M^+ \cap M^- = E$ (Theorem 2.6) and $m = m^*$ on E (Corollary 2.7), the real valuedness of q on $] -1, 1[$, hence on E , implies that f is well defined. Also, the continuity of q on M^+ implies the continuity of $m \mapsto \overline{q(m^*)}$ whenever $m \in M^-$. In fact, let m_α be a net in M^- converging to m . Then $m_\alpha^* = \tau(m_\alpha)$ converges to $\tau(m) = m^*$ by Lemma 1.2. Hence, using Corollary 2.7(1),

$$\overline{q(m_\alpha^*)} \rightarrow \overline{q(m^*)}.$$

Thus f is continuous on $M(H^\infty)$. Since for $a \in \mathbb{D}$, $(\varphi_a)^* = \varphi_{\bar{a}}$, we obtain that $f \in C(M(H^\infty))_{\text{sym}}$. \square

Let A be a commutative unital (real or complex) Banach algebra with unit element denoted by 1. The set of invertible n -tuples in A is the set

$$U_n(A) = \{(f_1, \dots, f_n) \in A^n \mid \exists g = (g_1, \dots, g_n) \in A^n : \sum_{j=1}^n f_j g_j = 1\}.$$

An element $(f_1, \dots, f_n, g) \in U_{n+1}(A)$ is said to be *reducible*, if there exists $(x_1, \dots, x_n) \in A^n$ so that

$$(f_1 + x_1 g, \dots, f_n + x_n g) \in U_n(A).$$

The smallest integer n for which every element in $U_{n+1}(A)$ is reducible is called the *Bass stable rank* of A and is denoted by $\text{bsr}(A)$. If no such integer exists, then $\text{bsr}(A) = \infty$.

A related concept is that of the *topological stable rank*, $\text{tsr}(A)$, of A (see [23]). This is the smallest integer n such that $U_n(A)$ is dense in A^n . If no such n exists, then $\text{tsr}(A) = \infty$. It is well known that $\text{bsr}(A) \leq \text{tsr}(A)$ (see [23, 21]).

Many papers have dealt with the determination of the Bass and/or topological stable rank for concrete function algebras (see for instance [3, 4, 5, 11, 20, 24, 25, 26, 27, 28, 30, 31]). It has been shown by Vaserstein [32] and Rieffel [23] that whenever X is a compact Hausdorff space, then

$$\text{tsr}(C(X, \mathbb{C})) = \text{bsr}(C(X, \mathbb{C})) = \left\lfloor \frac{\dim X}{2} \right\rfloor + 1$$

and

$$\text{tsr}(C(X, \mathbb{R})) = \text{bsr}(C(X, \mathbb{R})) = \dim X + 1.$$

The following result can now be deduced from Theorem 3.2 and Suárez's result [29] that the covering dimension of $M(H^\infty)$ is 2.

Corollary 4.2.

- (1) $\text{tsr } C(M(H^\infty)) = \text{bsr } C(M(H^\infty)) = 2;$
- (2) $\text{tsr } C(E, \mathbb{R}) = \text{bsr } C(E, \mathbb{R}) = 2;$
- (3) $\text{tsr } C(M^+, \mathbb{C}) = \text{bsr } C(M^+, \mathbb{C}) = 2.$

For an n -tuple $\mathbf{f} = (f_1, \dots, f_n)$ of complex-valued functions, let

$$|\mathbf{f}| = \left(\sum_{j=1}^n |f_j|^2 \right)^{1/2}.$$

As usual, S^n denotes the unit sphere

$$\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1\}$$

in \mathbb{R}^{n+1} . Finally, if $(z_1, z_2) \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$, then we say that $(z_1, z_2) \in S^3$.

We are now able to prove the main result of this paper. For matter of comparison, recall that $\text{bsr}(H^\infty) = 1$ ([31]), $\text{tsr}(H^\infty) = 2$ ([30]) and $\text{bsr}(H_\mathbb{R}^\infty) = \text{tsr}(H_\mathbb{R}^\infty) = 2$ ([20]).

Theorem 4.3. $\text{tsr } C(M(H^\infty))_{\text{sym}} = \text{bsr } C(M(H^\infty))_{\text{sym}} = 2$.

Proof. This follows as in [16] by using that $\text{bsr } C(E, \mathbb{R}) = 2$ and $\text{bsr } C(M^+, \mathbb{C}) = 2$. For the reader's convenience we present those parts that need replacing \mathbb{D} by $M(H^\infty)$, and \mathbb{D}^+ by M^+ .

1. We first note that $\text{bsr}(C(M(H^\infty))_{\text{sym}}) > 1$, since the invertible pair $(z, 1 - z^2)$ is not reducible.

2. Next we indicate how to prove that $\text{tsr}(C(M(H^\infty))_{\text{sym}}) \leq 2$. Let $\mathbf{f} = (f_1, f_2) \in (C(M(H^\infty))_{\text{sym}})^2$ and

$$E_n = \{m \in M^+ : |\mathbf{f}(m)| \geq 1/n\}.$$

Step 1 Suppose that $E_n \cap E \neq \emptyset$. We claim that there is an \mathbb{R}^2 -valued extension of the tuple $\mathbf{f}/|\mathbf{f}| \in C(E_n \cap E, S^1)$ to $\tilde{\mathbf{f}}_n \in C(E, S^1)$.

To prove this, we choose $g_n \in C(E, \mathbb{R})$ with $g_n \equiv 0$ on $E_n \cap E$ and $g_n \equiv 1$ on $Z(f_1) \cap Z(f_2) \cap E$ (Urysohn's Lemma).

Then the triple (f_1, f_2, g_n) is invertible in $C(E, \mathbb{R})$. Since by Corollary 4.2 $\text{bsr}(C(E, \mathbb{R})) = 2$, there exist $h_{1,n}, h_{2,n} \in C(E, \mathbb{R})$ such that

$$(f_1 + h_{1,n}g_n, f_2 + h_{2,n}g_n)$$

is invertible in $C(E, \mathbb{R})$. Now the pair

$$\tilde{\mathbf{f}}_n := (f_1 + h_{1,n}g_n, f_2 + h_{2,n}g_n) / |(f_1 + h_{1,n}g_n, f_2 + h_{2,n}g_n)|$$

is the desired extension. We point out that $\tilde{\mathbf{f}}_n$ is \mathbb{R}^2 -valued.

If $E_n \cap E = \emptyset$, then we let $\tilde{\mathbf{f}}_n = (1, 0)$.

Step 2 Next we claim that there exists a \mathbb{C}^2 -valued extension of $\mathbf{f}/|\mathbf{f}| \in C(E_n, S^3)$ to $\hat{\mathbf{f}}_n \in C(M^+, S^3)$ that coincides on E with $\tilde{\mathbf{f}}_n$.

In fact, define $\mathbf{F}_n = (F_{1,n}, F_{2,n})$ by

$$(4.1) \quad \mathbf{F}_n(m) = \mathbf{f}(m)/|\mathbf{f}(m)| \text{ whenever } m \in E_n,$$

$$(4.2) \quad \mathbf{F}_n(m) = \tilde{\mathbf{f}}_n(m) \text{ whenever } m \in E$$

and extended continuously to $M(H^\infty)$ by Tietze. Note that \mathbf{F}_n is well defined, due to Step 1. Now let $G_n \in C(M^+, \mathbb{R})$ be a real valued continuous function with $G_n \equiv 0$ on $E_n \cup E$ and $G_n \equiv 1$ on $Z(F_{1,n}) \cap Z(F_{2,n})$. Then the triple $(F_{1,n}, F_{2,n}, G_n)$ is invertible in $C(M^+, \mathbb{C})$. Since by

Corollary 4.2 $\text{bsr}(C(M^+, \mathbb{C})) = 2$, there exist $H_{1,n}, H_{2,n} \in C(M^+, \mathbb{C})$ such that

$$(F_{1,n} + H_{1,n}G_n, F_{2,n} + H_{2,n}G_n)$$

is invertible in $C(M^+, \mathbb{C})$. Now the pair

$$\hat{\mathbf{f}}_n = (F_{1,n} + H_{1,n}G_n, F_{2,n} + H_{2,n}G_n) / |(F_{1,n} + H_{1,n}G_n, F_{2,n} + H_{2,n}G_n)|$$

is the desired extension.

Step 3 It is easy to check that $|\mathbf{f} - (|\mathbf{f}| + 1/n)\hat{\mathbf{f}}_n| \leq 3/n$ on M^+ .

Step 4 In the steps above we have found a \mathbb{C}^2 -valued function

$$\mathbf{g}_n := (|\mathbf{f}| + 1/n)\hat{\mathbf{f}}_n$$

with $|\mathbf{f} - \mathbf{g}_n| \leq 3/n$ on M^+ . Note that \mathbf{g}_n is \mathbb{R}^2 -valued on $E \supseteq]-1, 1[$. Thus by Lemma 4.1 we can use reflection to define a \mathbb{C}^2 -valued function Φ_n on M (whose components are in $C(M(H^\infty))_{\text{sym}}$) so that $|\mathbf{f} - \Phi_n| \leq 3/n$ on $M(H^\infty)$ and such that $|\Phi_n| \geq \frac{1}{n} > 0$ on $M(H^\infty)$. \square

It remains an open problem which pairs (f, g) of functions in $C(M(H^\infty))_{\text{sym}}$ are reducible. Recall that in $H_\mathbb{R}^\infty$ an invertible pair (f, g) is reducible if and only if f has constant sign on the set $Z(g) \cap E$ (see [34, 35] and [14]). The situation in $C(M(H^\infty))_{\text{sym}}$ is more difficult, since a) the behaviour of f outside E is not determined by that in E (in contrast to the analytic case) and b) the Bass stable rank of $C(M(H^\infty))$ is two, and not one. So a characterization of the reducible elements in $C(M(H^\infty))_{\text{sym}}$ must also involve conditions outside $M(H^\infty) \setminus E$. A necessary condition for example is the following:

Suppose that (f, g) is reducible, say $u = f + hg \neq 0$ on $M(H^\infty)$ and let C be a connected component of $M(H^\infty) \setminus Z(g)$. Suppose that the closure of C is contained in \mathbb{D} . Then u is a zero free (continuous) extension of $f|_{\partial C}$ to C . Thus the Brouwer degree of f satisfies $d(f, C, 0) = d(u, C, 0) = 0$.

Necessary and sufficient criteria for reducibility of individual pairs in $C(K)$ and $C(K)_{\text{sym}}$, where $K \subseteq C$ is compact, have meanwhile been developed (see [26] for preliminary material and [18] for a full solution).

5. CONJECTURE

In view of the results in this paper and the ones in [17], we conjecture that the following is true:

Conjecture. *Let X be a compact Hausdorff space, and τ a topological involution of X . Denote the set of fixed points of τ by E . Then*

$$\text{bsr } C(X, \tau) = \text{tsr } C(X, \tau) = \max \left\{ \left\lfloor \frac{\dim X}{2} \right\rfloor, \dim E \right\} + 1.$$

Note added in proof

Meanwhile this conjecture has been confirmed (see [19]).

Acknowledgements I thank Rudolf Rupp for his contributions to joint work preceding this paper, without those Theorem 4.3 would not have come to live. I also thank Amol Sasane and Brett Wick for several discussions concerning contractability or non-contractability of the spectrum of H^∞ .

REFERENCES

- [1] S. Axler, P. Gorkin. Sequences in the maximal ideal space of H^∞ , Proc. Amer. Math. Soc. 108 (1990), 731-740.
- [2] Ch. Bishop. Some characterizations of $C(\mathcal{M})$, Proc. Amer. Math. Soc. 124 (1996), 2695-2701.
- [3] G. Corach, A. Larotonda. Stable range in Banach algebras, J. Pure and Appl. Algebra 32 (1984), 289-300.
- [4] G. Corach, F.D. Suárez. Stable rank in holomorphic function algebras, Illinois J. Math. 29 (1985), 627-639.
- [5] G. Corach, F.D. Suárez. On the stable range of uniform algebras and H^∞ . Proc. Amer. Math. Soc. 98 (1986), 607-610.
- [6] R. Engelking. *Dimension Theory*, North Holland Publ. Comp., Amsterdam, 1978.
- [7] T.W. Gamelin. *Uniform algebras*, Chelsea Pub. Company, New York 1984.
- [8] J.B. Garnett. *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [9] K. Hoffman. Bounded analytic functions and Gleason parts, Ann. of Math. (2) 86 (1967), 74-111.
- [10] K. Izuchi. Common zero sets of equivalent singular inner functions II, Studia Math. 180 (2007), 133-142.
- [11] P. W. Jones, D. Marshall, T. Wolff. Stable rank of the disc algebra, Proc. Amer. Math. Soc. 96 (1986), 603-604.
- [12] S.H. Kulkarni, B.V. Limaye. *Real Function algebras*, Marcel Dekker, New York, 1992.
- [13] R. Mortini. A distinguished real Banach algebra. Proc. Indian Acad. Sci. (Math. Sci.) 119 (2009), 629-634.
- [14] R. Mortini. Reducibility of function pairs in $H^\infty_{\mathbb{R}}$, to appear in Algebra i Analiz resp. St. Petersburg Math J.
- [15] R. Mortini, A. Nicolau. Frostman shifts of inner functions. J. d'Analyse Math. 92 (2004), 285-326.
- [16] R. Mortini, R. Rupp. Approximation by invertible elements and the generalized E -stable rank for $A(\mathbf{D})_{\mathbb{R}}$ and $C(\mathbf{D})_{\text{sym}}$, Math. Scand. 109 (2011), 114-132.
- [17] R. Mortini, R. Rupp. Real-symmetric extensions of invertible tuples of multivariable continuous functions, to appear in Complex Analysis and Operator Theory.
- [18] R. Mortini, R. Rupp. The Bass stable rank for the real Banach algebra $A(K)_{\text{sym}}$, J. Funct. Analysis 261 (2011), 2214-2237.
- [19] R. Mortini, R. Rupp. Stable rank for the real function algebra $C(X, \tau)$, to appear in Indiana Univ. Math. J.

- [20] R. Mortini, B. Wick. The Bass and topological stable ranks of $H_{\mathbb{R}}^{\infty}(\mathbf{D})$ and $A_{\mathbb{R}}(\mathbf{D})$, J. Reine Angew. Math. 636 (2009), 175-191.
- [21] R. Mortini, B. Wick. Spectral characteristics and stable ranks for the Sarason algebra $H^{\infty} + C$, Michigan Math. J. 59 (2010), 395-409.
- [22] A.R. Pears, *Dimension theory of general spaces*, Cambridge Univ. Press London, New York, Melbourne, 1975.
- [23] M. Rieffel. Dimension and stable rank in the K -theory of C^* -algebras, Proc. London Math. Soc. 46 (1983), 301-333.
- [24] R. Rupp. Stable ranks of subalgebras of the disc algebra, Proc. Amer. Math. Soc. 108 (1990), 137-142.
- [25] R. Rupp. Stable rank of finitely generated algebras, Archiv Math. 55 (1990), 438-444.
- [26] R. Rupp. Stable rank and boundary principle, Topology Appl. 40 (1991), 307-316.
- [27] R. Rupp, A. Sasane. On the stable rank and reducibility in algebras of real symmetric functions, Math. Nachrichten. 283 (2010), 1194-1206.
- [28] R. Rupp, A. Sasane. Reducibility in $A_{\mathbb{R}}(K)$, $C_{\mathbb{R}}(K)$ and $A(K)$, Canad. J. Math. 62 (2010), 646-667.
- [29] D. Suárez. Čech cohomology and covering dimension for the H^{∞} maximal ideal space, J. Funct. Anal. 123 (1994), 233-263.
- [30] D. Suárez. Trivial Gleason parts and the topological stable rank of H^{∞} , Amer. J. Math. 118 (1996), 879-904.
- [31] S. Treil. The stable rank of H^{∞} equals 1, J. Funct. Anal. 109 (1992), 130-154.
- [32] L. Vasershtein. Stable rank of rings and dimensionality of topological spaces, Funct. Anal. Appl. 5 (1971), 102-110; translation from Funkts. Anal. Prilozh. 5 (1971), No.2, 17-27.
- [33] B. Wick. A note about stabilization in $A_{\mathbb{R}}(\mathbb{D})$, Math. Nachrichten 282 (2009), 912-916.
- [34] B. Wick. Stabilization in $H_{\mathbb{R}}^{\infty}(\mathbb{D})$, Publ. Mat. 54 (2010), 25-52.
- [35] B. Wick. Corrigenda: "Stabilization in $H_{\mathbb{R}}^{\infty}(\mathbb{D})$ " Publ. Mat. 55 (2011), 251-260.

DÉPARTEMENT DE MATHÉMATIQUES, LMAM, UMR 7122, UNIVERSITÉ PAUL VERLAINE, ÎLE DU SAULCY, F-57045 METZ, FRANCE

E-mail address: mortini@univ-metz.fr